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HYDRAULICS

CHARACTERISTIC FEATURES OF A LIQUID FLOW IN AN OPEN CHANNEL

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1. Saint-Venant classified streams in open channels into two groups: rivers or «tranquil» flows, and torrents or «rapid» flows, connecting this classification ([1]) with the propagation of disturbances over the free surface of the liquid in the open channel. In tranquil flows ($U_0 < \sqrt{gH_0}$) the disturbances may propagate downstream as well as up. In rapid flows ($U_0 > \sqrt{gH_0}$) disturbances do not propagate upstream. To determine the kind of flow, Saint-Venant proposed the following number*([1]):

$$\frac{U_0^2}{gH_0}\left(\text{or } U_0^2 \,/\, g\frac{F_0}{B_0}\right). \tag{1}$$

When this number equals unity, the corresponding values of U_0 and H_0 (and of the slope of the bottom i_0) are called critical. The value of this number in open channel hydraulics is well known.

2. Observations on structures and theoretical investigations show that Saint-Venant's criterion and his classification of flows into tranquil and rapid ones is not sufficient; a supplementary classification is needed, as well as a criterion to characterize a disturbance as progressing along the stream with reference to its variation in form and to its action on the stream, *i. e.*, the possibility of steady motion in the stream. Boussinesq([2]) has pointed out that, taking account of the curvature of filaments (which thus far has been possible only in the plane problem, *i. e.*, in a broad rectangular channel), one will find that the friction and the slope combine to decrease the velocity of propagation of individual parts of the wave disturbance, as long as (with $C=\text{const}$) $U_0^2 < 4gH_0$.

In the following we shall make use of Saint-Venant's equations for gradually varying unsteady flow in prismatic channels and also of equations which we have published previously ([3]), employing the same notations.

Analysis of the change in form of a disturbance, when moving along a stream, gives grounds for introducing the number**

$$\frac{U_0^2M_0^2(1+\beta)^2}{p^2gF_0/B_0} \quad \text{or} \quad \frac{U_0M_0(1+\beta)}{p(W_0-U_0)} \tag{2}$$

as a criterion of the possibility of steady uniform flow (or of its stabi

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* This form was given later. Number (1) is more commonly known as Froude's number.

** Number (2) in its second form is independent of the method by which the influence of non-uniformity in the velocity distribution on the velocity of the wave front propagation is evaluated. The subscript 0 denotes the hydraulic elements of uniform motion.

lity). If this number is less than unity, the flow is tranquil or rapid; if it is greater than unity, the flow is said to be «ultra-rapid» («wavy flow»). The values of U_0, H_0 and the slope of the bottom i_0 are described as «second» critical (*e. g.* U^{II}_{cr}) when number (2) is equal to unity.

Determining the variation in the velocity ω of propagation of the level (depth or area) along the free surface of the wave disturbing the uniform flow ($dF/dt=0$), we get at its front

$$\frac{\partial\omega}{\partial F}=-\frac{1}{r^2}\frac{dr}{dt}=\frac{\partial W}{\partial F}+g\frac{i_0}{rU_0}\left[\frac{p}{2}-\frac{M_0(1+\beta)U_0}{2(W_0-U_0)}\right]=\frac{\partial W}{\partial F}+\frac{E_0}{r} \qquad (3)$$

where p is the exponent in the law of resistance*; $\partial W/\partial F=$ $=\frac{3}{2}\frac{W_0-U_0}{F_0}N$ is the change in the velocity W (of propagation of the level) along the same surface in the case of a perfect fluid. From (3) with a small height h we have $\omega=W+E_0 s$, where s is measured downstream from the front of the disturbances. For laminary flow in a broad rectangular channel, $p=1$, $\beta=1$, $M=1$, and the formula for number (2) becomes

$$4U_0^2\,/\,gH_0 \qquad (4)$$

It follows from Saint-Venant's equations, and has been demonstrated by Saint-Venant himself, that the waves of a perfect fluid ($i_0=0$, $U^2\,/\,C^2R=0$) propagate without any changes in height (there being no losses of energy) and are subject only to shear in the direction parallel to the initial free surface. Therefore the relation $\partial\omega/\partial F \gtrless \partial W/\partial F$ determines the damping of the disturbance and, consequently, the stability of the initial motion. For example, if $\partial\omega\,/\,\partial F<\partial W\,/\,\partial F$ for a descending positive wave ($r<0$) of a real fluid, then, observing a certain level h above the original surface; it will be seen that during a length of time dt a volume $(\partial W/\partial F-\partial\omega/\partial F)\frac{h^2}{2}Bdt$ is displaced from the part of the wave above this level into the part below. Referring this volume to the original volume of a small wave at the front in the shape of an elementary prism of height a, we get an estimate of the velocity of damping of the disturbance (in the form of a positive or negative wave) at the beginning of the wave propagation $dh/dt=r(\partial W/\partial F-\partial\omega/\partial F)\,h=$ $=-E_0 h$ (and the velocity of lengthening of its base, whose initial length was l, as deduced from the condition for the invariability of the volume of the disturbance $dl/dt=E_0 l$, *i. e.* the velocity of propagation of the rear part of the wave at the original surface level $\omega\approx W_0-E_0 l$).

Similarly, for a small rate of flow q of the wave, we have $dq/dt=$ $=-E_0 q$.

The quantity $2\frac{E_0}{g}\frac{W_0-U_0}{F_0/B_0}h$ determines the change (its first term) of the resisting force per unit weight of the fluid at a small wave height h, as compared to this force for the initial uniform motion.

If the flow is tranquil or rapid, positive or negative waves, when moving along the stream, decrease in height, dampen, and the original steady flow is restored. Negative waves always flatten. Positive waves may become steeper and break down only if the wave forms quickly, but even in this case, when, for instance, for the descending wave $\partial\omega/\partial F>0$, the inequality $\partial\omega/\partial F<\partial W/\partial F$ remains true.

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* If p is assumed to be always 2, then C will, in general, be also a function of Reynolds number. Whatever the formula used to determine C, we have $\beta=\frac{2R}{C}\frac{C}{dR}$.

When number (2) equals unity (and $E_0 = 0$), we have a critical case — positive and negative waves retain their heights and rates of flow; negative waves flatten, while positive waves grow steeper and break down.

When number (2) is greater than unity, the flow is ultra-rapid. Positive waves grow steeper and break down. Negative waves, if formed rapidly, flatten, but if formed slowly, become steeper and in both cases finally tend to move along without changing their shape ($\partial\omega/\partial F \to 0$). Disturbances (positive and negative waves) in an ultra-rapid stream grow as they move along the stream (the body of the wave moving along more quickly than do wave bodies in a perfect fluid). Therefore, steady uniform flow is impossible (with constant depth and velocity of flow in a given section) in an ultra-rapid flow, and the motion of such a flow is always wavy.

3. It should be mentioned that number (2) can be used to characterize the progress along a stream of «disturbances» and the stability of the steady flow only if the motion is known beforehand to be «uniform». Therefore, in order to generalize the above to steady non-uniform moti n ($dF/dt \neq 0$, $dQ/dt = 0$) we find the variation of the velocity ω' with which the heights h or areas f of the disturbance move along its free surface, parallel to the latter (and not to the bottom, as was the case in equation (3)). Denoting $r' = r - dF/ds$, and applying equation (16) from a previous paper ([3]), we find

$$\frac{\partial\omega'}{\partial f} = -\frac{1}{r'^2}\frac{dr'}{dt} = \frac{3}{2}\frac{W-U}{F}N + \frac{1}{r'}\left\{E - \frac{1}{4F}\frac{dF}{ds}\left[(12U - 3W) + \right.\right.$$

$$\left.\left. + (5W - 4U)\frac{F}{B}\frac{dB}{dF}\right]\right\} = \frac{\partial W}{\partial f} + \frac{V}{r'} \qquad (5)$$

where $E = g\frac{i_f}{U}\left[\frac{p}{2} - \frac{M}{2}\frac{(1+\beta)U}{(W-U)}\right]$; i_f is the friction slope; and $\partial W/\partial f$ is the variation of the velocity W of a perfect fluid. When $V < 0$, the shape change characteristic is the same as for an ultra-rapid flow, and when $V > 0$, it is the same as for tranquil and rapid flows. But equation (5) can be used to analyse the propagation of disturbances only in the region where the initial non-uniform motion varies very gradually (so gradually, indeed, that within a sufficiently long wave-length the curvature of the initial free surface can be neglected), *i. e.* in regions where the deviation from steady flow is small. Therefore we shall use equation (5) to solve the problem of non-uniform motion, in the form of a falling surface curve ($dF/ds < 0$) * in a chute, or, generally, in a steep slope with $i_0 > i_{cr}^{II}$ (number (2) greater than unity), since it is known beforehand that this kind of water motion asymptotically approaches the motion in the shape of an ultra-rapid flow.

In the region where the movement is nearly «uniform» flow, the curvature of the free surface is small, and we have $dh/dt = -Vh$ for the height of a small disturbance over its short path. Thus, the criterion in this case is

$$\frac{M(1+\beta)U}{p(W-U)} + \frac{U}{gi_f}\frac{1}{2pF}\frac{dF}{ds}\left[(12U - 3W) + (5W - 4U)\frac{F}{B}\frac{dB}{dF}\right] \gtrless 1 \qquad (6)$$

As long as this number is less than unity, the flow is rapid. Below the critical section, in which this number is equal to unity, the number becomes higher than unity, the flow is ultra-rapid, steady flow becomes impossible, waves begin to grow and in a certain section, lying below the «second» critical

* Other cases of non-uniform movement are to be considered in a later part of this investigation.

section, the flow becomes pronouncedly wavy. Thus, the possibility of the flow passing into the ultra-rapid state in a given structure is determined by the length of the falling surface as well.

If the length L of the structure is less than the critical length determined in this manner L_{cr}, the movement in such a structure is steady and can be computed by the usual formulae for non-uniform flow.

For the back-water curve on a slope of the same kind, when $H < H_{cr}^{II}$, steady flow is, generally, unstable, at least in those sections where the deviations from uniform movement are small; the stream is ultra-rapid, and therefore in these sections the equations of non-uniform steady flow can no longer be applied. The back-water curve is a diffusor.

4. The above determines the limits* of applicability of the equations of open channel hydraulics and is of considerable practical importance, *e. g.* in the computation of chutes, jet vibrations in dam spillways, shaft spillways, *etc*. If necessary, numbers (2) and (6) can be used in calculations intended to prevent ultra-rapid flow by the construction of walls along the flow, which will separate it into jets thereby changing the values of U and M.

Analysis of the experiments of Hopf (4), Zhavoronkov (5) and other authors on laminar motion in thin layers shows that a laminar flow can become ultra-rapid (or wavy flow); in first approximation the state of the flow in this case can be estimated by number (4) [and number (6)]. This indicates that passing to the ultra-rapid flow is not determined by turbulent pulsation.

5. The accuracy of the above statements corresponds to that of Saint-Venant and Belanger's equations. For further refinement it is necessary to take into account the curvature of the wave surfaces, and, in the case of laminar flow at least, the action of surface tension, as well as the velocity distribution over the section and the shearing stresses along the wetted perimeter (especially in channels with gently sloping sides).

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* It is interesting to note that Ralph Powell, in analysing his experiments on the resistance to flow in open smooth channels, which were published not long ago (6), writes that «it is surprising» that the law of resistance should change so abruptly and «it is also surprising that the change...» came between Fr=1.69 and Fr=2.49 instead of at Fr=1. This «surprising» change in the law of resistance can easily be explained on the basis of the present paper, and is a good experimental confirmation of our theses. It is clear that this change should be connected with number (2), *i. e.* with the transition from rapid to ultra-rapid flow, rather than with Fr=1 (*i.e.*, with the transition from tranquil to rapid flow).

Indeed, number (2) for the experiment in which $Fr = U_0 \sqrt{gH_0} = 1.69$, equals $(0.96)^2$, and for the experiment in which Fr=2.49 it equals $(1.26)^2$ (our analysis of Powell's experiments for channels with smooth walls showed that the values of C observed for them fit in well with Lea's formula, up to the change in the law of resistance; $p=1.75$ and $\beta = 0.25$). Moreover, with $Fr \geqslant 2.49$ Powell noted that waves begin to appear, which should be characteristic of the «ultra-rapid» stream mentioned above. Powell did not carry out any experiments between Fr=1.69 and Fr=2.49.